Ex 1 Let $f(x) = e^x$. Find the degree k polynomial, $g_k(x)$, which has $g_k(0) = f(0)$, $g_k'(0) = f'(0)$, $g_k''(0) = f''(0)$,..., $g_k^{(k)}(0) = f^{(k)}(0)$.

- $g_0(x) = a_0$, need $g_0(0) = f(0) = 1$
  ↳ $g_0(x) = 1$ $\quad a_0$

- $g_1(x) = a_0 + a_1 x$
  ↳ need $g_1(0) = f(0) = 1$
  $\quad a_0$
  ↳ need $g_1'(0) = f'(0) = 1$
  $\quad a_1$
  $g_1'(x) = a_1 \longrightarrow g_1(x) = 1 + x$

- $g_2(x) = a_0 + a_1 x + a_2 x^2$
  ↳ need $a_0 = g_2(0) = f(0) = 1$
  ↳ need $a_1 = g_2'(0) = f'(0) = 1$
  $- g_2'(x) = a_1 + 2a_2 x$
  ↳ need $2a_2 = g_2''(0) = f''(0) = 1 \rightarrow a_2 = \frac{1}{2}$
  $- g_2''(x) = 2a_2$ $\quad \Downarrow$ $g_2(x) = 1 + x + \frac{x^2}{2}$

  ⋮

- $g_k(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^k}{k!}$

Ex 2 Let $f(x) = e^x$. Find the degree k polynomial, $h_k(x)$, which has $h_k(5) = f(5)$, $h_k'(5) = f'(5)$, $h_k''(5) = f''(5)$,..., $h_k^{(k)}(5) = f^{(k)}(5)$.

- $h_0(x) = b_0$, need $h_0(5) = f(5) = e^5$
  $\| \quad \| \quad \longrightarrow h_0(x) = e^5$
  b_0

- $h_1(x) = b + mx$, need $h_1(5) = f(5) = e^5$
  $\| \quad \longrightarrow b = -4e^5$
  $b + 5m = b + 5e^5$
  $h_1'(x) = m$
  $h_1'(5) = f'(5) = e^5$
  $\|$
  m

$b + 5e^5 = e^5 \longrightarrow b = e^5 - 5e^5 = e^5(1-5)$
$= -4e^5$

$\hookrightarrow h_1(x) = -4e^5 + e^5 \cdot x$

- $h_1(x) = b_0 + b_1(x-5) \longrightarrow h_1'(x) = b_1$
  need: $b_0 = h_1(5) = f(5) = e^5$
  need: $b_1 = h_1'(5) = f'(5) = e^5$

$\hookrightarrow h_1(x) = e^5 + e^5(x-5)$

- $h_2(x) = b_0 + b_1(x-5) + b_2(x-5)^2$ $\quad h_2'(x) = b_1 + 2b_2(x-5)$, $h_2''(x) = 2b_2$
  need: $b_0 = h_2(5) = f(5) = e^5$
  $b_1 = h_2'(5) = f'(5) = e^5$

$$2b_2 = h_2''(5) = f''(5) = e^5 \rightarrow b_2 = \frac{e^5}{2}$$

$$\hookrightarrow h_2(x) = e^5 + e^5(x-5) + \frac{e^5}{2}\cdot(x-5)^2$$

$\vdots$

- $$h_k(x) = e^5 + e^5(x-5) + \frac{e^5}{2}\cdot(x-5)^2 + \cdots + \frac{e^5}{k!}(x-5)^k$$

Q. What if we take k to infinity?

Note: $1 + x + \frac{x^2}{2} + \cdots + \frac{x^k}{k!}$ is the k^{th} partial sum of $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

So far: $\sum_{n=0}^{\infty} \frac{x^n}{n!} \approx e^x$ for x "near" 0

$\sum_{n=0}^{\infty} \frac{e^5 \cdot (x-5)^n}{n!} \approx e^x$ for x "near" 5

Questions:

(1) For which x does $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converge?

(2) If $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges, what does it converge to?

Def: A series of the form $\sum_{n=0}^{\infty} c_n(x-a)^n$ is called a power series

Q. How do you want to think about power series?

(1) As a function of x

(2) Fix x, think about power series as a number

Key Skill:

(1) See x as variable when
- deriving
- integrating

(2) See x as a constant when
- computing values
- determining convergence

Def: For the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$...

- The constant "a" is the center
- The numbers $c_0, c_1, c_2, \ldots$ are the coefficients

Q. For which values of x does a power series converge?

↳ think about x as a number

Ex 3 For which values of x does the power series $\sum_{n=0}^{\infty} x^n$ converge?

$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$ (geometric) $= \frac{1}{1-x}$

$\sum_{n=1}^{\infty} \frac{1}{n^x}$ ← x-series

the set of real numbers $\downarrow$ $\mathbb{R}$

$\sum_{n=0}^{\infty} x^n$ converges if and only if $|x| < 1$ iff $x \in (-1, 1)$

Ex 4 For which values of x does the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converge?

Fix $x \in \mathbb{R}$ ← x is in the set of real numbers

R.T: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right|$

$= \lim_{n\to\infty} \left|\frac{x}{n+1}\right| = 0 < 1$ ← no matter what x is

$\Rightarrow \sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges no matter what x is, i.e. for all $x \in (-\infty, \infty)$

Ex 5 For which values of x does the power series $\sum_{n=0}^{\infty} n! \cdot x^n$ converge?

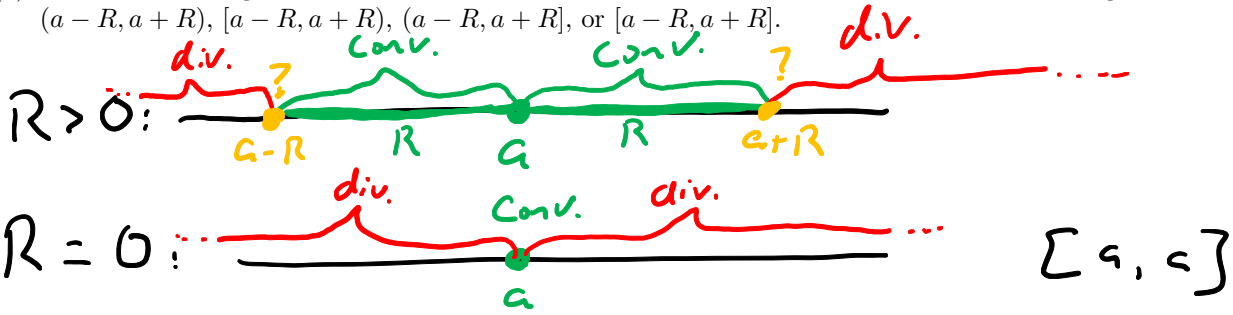
R.T: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!\, x^{n+1}}{n!\, x^n}\right|$

$= \lim_{n\to\infty} |(n+1) \cdot x| = \begin{cases} \infty & x \neq 0 \\ 0 & x = 0 \end{cases}$

$\Rightarrow \sum_{n=0}^{\infty} n! x^n$ converges if and only if $x = 0 \leftrightarrow x \in [0, 0]$

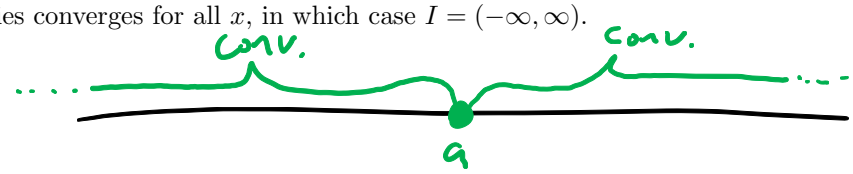
Thm: If I is the set of all x-values for which $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges, there are two possibilities.

(a) The series converges on an interval of "radius" $R \geqslant 0$, in which case I is one of the following: $(a-R, a+R)$, $[a-R, a+R)$, $(a-R, a+R]$, or $[a-R, a+R]$.



[a, c]

(b) The series converges for all x, in which case $I = (-\infty, \infty)$.



Def: In the setting of the above theorem,

- I is called the interval of convergence
- R is called the radius of convergence

Q. What is R if $I = (-\infty, \infty)$?

We say "$R = \infty$"

Q. Given a power series, how do we find the radius and interval of convergence?

↳ use the ratio / root test

Ex 6 Find the center, radius of convergence, and interval of convergence for $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}5^n}(x-3)^n$.

The center is...

(a) -5 (c) -1 (e) 0.5 (g) 3

(b) -2 (d) 0 (f) 1 (h) 5

R.T.: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r$

$r < 1 \longrightarrow$ convergence

$r > 1 \longrightarrow$ divergence

$r = 1 \longrightarrow$???

$$r = \lim_{n\to\infty} \left| \frac{(x-3)^{n+1}}{\sqrt{n+1}\,5^{n+1}} \cdot \frac{\sqrt{n}\,5^n}{(x-3)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(x-3)}{5} \cdot \sqrt{\frac{n}{n+1}} \right| = \left|\frac{x-3}{5}\right| \cdot \lim_{n\to\infty} \sqrt{\frac{n}{n+1}}$$

$$= \left|\frac{x-3}{5}\right| \cdot \sqrt{\lim_{n\to\infty} \frac{n}{n+1}} = \left|\frac{x-3}{5}\right| \sqrt{1} = \left|\frac{x-3}{5}\right|$$

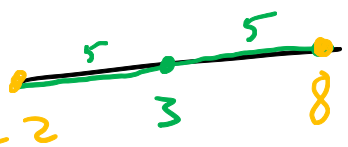
$\left|\frac{x-3}{5}\right| < 1 \longrightarrow$ convergence

$\updownarrow$

$|x-3| < 5 \iff x$ is within 5 of 3

$\iff x \in (3-5, 3+5)$

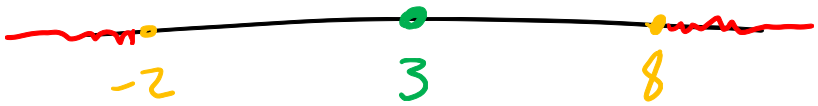
$\in (-2, 8)$



$\left|\frac{x-3}{5}\right| > 1 \longrightarrow$ divergence

$\updownarrow$

$|x-3| > 5 \longleftrightarrow x$ is more than 5 away from 3 $\longleftrightarrow x \in (-\infty, -2) \cup (8, \infty)$



$\left|\frac{x-3}{5}\right| = 1 \longrightarrow ???$

$\updownarrow$

$|x-3| = 5 \longleftrightarrow x$ is exactly 5 away from 3 $\longleftrightarrow x = 8$ or $x = -2$

Check: $x = 8$

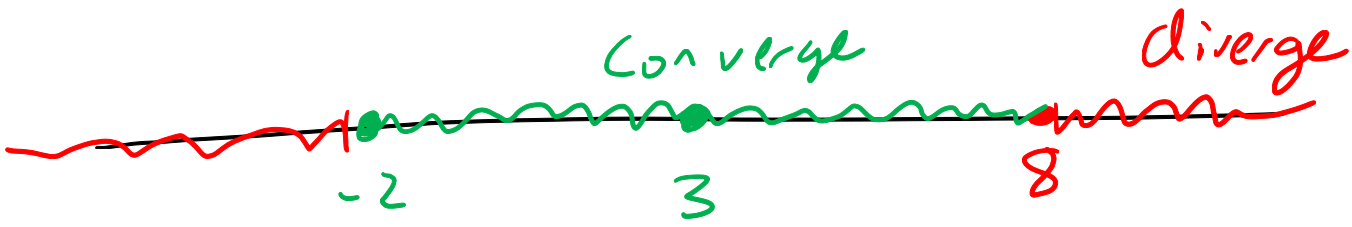
$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} \cdot 5^n} \cdot (x-3)^n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n} \cdot 5^n} \cdot 5^n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

which diverges by the p-test $\left(p = \frac{1}{2}\right)$

Check: $x = -2$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} \cdot 5^n} (x-3)^n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n} \cdot 5^n} (-5)^n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left(\frac{-5}{5}\right)^n$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$ which converges by the A.S.T.



$$I = [-2, 8)$$

$$R = 5$$

Q. How do we find I in general?

(a) Compute the limit from the root or ratio test
↳ $\left|\frac{x-a}{c}\right|$ or ∞ or 0

(b) Find the the interval of x-values to make that limit less than 1.

(c) Check endpoints using Ch. 5 series tests.

Ex 7 Find the interval of convergence for $\sum_{n=0}^{\infty} n^n(x-3)^n$.

(a) R.T.: $\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|n^n\cdot(x-3)^n\right|^{1/n}$

$= \lim_{n\to\infty} |n\,(x-3)|$

$= \begin{cases} \infty > 1 & x \neq 3 \\ 0 < 1 & x = 3 \end{cases}$

(b) Interval of x-values to make $\uparrow < 1$: $[3,3] = I$

Center : 3

$R = 0$

Ex 8 Find the interval of convergence for $\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(-3)^k \cdot (2k+1)}$. The center is...

(a) -1 (b) 0 (c) 1 (d) 2

$$\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(-3)^k (2k+1)} = \frac{x}{1 \cdot 1} + \frac{x^3}{(-3)\cdot(3)} + \frac{x^5}{(-3)^2 \cdot 5} + \cdots$$

$$= 0 \cdot x^0 + x + 0 \cdot x^2 + \frac{x^3}{(-3)(3)} + 0 \cdot x^4 + \cdots$$

$$\lim_{k\to\infty} \left| \frac{x^{2(k+1)+1}}{(-3)^{k+1}(2(k+1)+1)} \cdot \frac{(-3)^k \cdot (2k+1)}{x^{2k+1}} \right|$$

$$= \lim_{k\to\infty} \left| \frac{x^{2k+3}}{x^{2k+1}} \cdot \frac{(-3)^k}{(-3)^{k+1}} \cdot \frac{2k+1}{2k+3} \right|$$

$$= \lim_{k\to\infty} \left| x^{(2k+3)-(2k+1)} \cdot \frac{1}{-3} \cdot \frac{2k+1}{2k+3} \right|$$

$$= \lim_{k\to\infty} \left| x^2 \cdot \frac{1}{-3} \cdot \frac{2k+1}{2k+3} \right| = \left| \frac{x^2}{-3} \right| \cdot \lim_{k\to\infty} \left| \frac{2k+1}{2k+3} \right|$$

$$= \frac{x^2}{3}$$

Goal: $\frac{x^2}{3} < 1 \longleftrightarrow x^2 < 3 \longleftrightarrow x < \sqrt{3} \longleftrightarrow x \in (-\infty, \sqrt{3})$

$|x| < \sqrt{3} \longleftrightarrow x \in (-\sqrt{3}, \sqrt{3})$

Check: $x = \sqrt{3}$

$$\sum_{k=0}^{\infty} \frac{(\sqrt{3})^{2k+1}}{(-3)^k \cdot (2k+1)} = \sum_{k=0}^{\infty} \frac{(\sqrt{3})^{2k} \cdot \sqrt{3}}{(-3)^k \cdot (2k+1)}$$

$$= \sum_{k=0}^{\infty} \frac{\left((\sqrt{3})^2\right)^k \cdot \sqrt{3}}{(-3)^k \cdot (2k+1)} = \sum_{k=0}^{\infty} \frac{3^k \cdot \sqrt{3}}{(-3)^k \cdot (2k+1)}$$

$$= \sum_{k=0}^{\infty} \left(\frac{3}{-3}\right)^k \cdot \frac{\sqrt{3}}{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k \cdot \sqrt{3}}{2k+1}$$

Converges by the A. S. T.

Check: $x = -\sqrt{3}$ (see ↑ for arithmetic details)

$$\sum_{k=0}^{\infty} \frac{(-\sqrt{3})^{2k+1}}{(-3)^k \cdot (2k+1)} = \sum_{k=0}^{\infty} \frac{\left((-\sqrt{3})^2\right)^k (-\sqrt{3})}{(-3)^k \cdot (2k+1)}$$

$$= \sum_{k=0}^{\infty} \frac{3^k}{(-3)^k} \cdot \frac{-\sqrt{3}}{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k(-\sqrt{3})}{2k+1}$$ which

converges by the A. S. T.

$$\Rightarrow \boxed{I = [-\sqrt{3}, \sqrt{3}]}$$

Ex 9 Find the interval of convergence for $\sum_{n=0}^{\infty} \frac{(2x+8)^n}{n!}$. The center is...

(a) -8 (c) -2 (e) 2 (g) 8
(b) -4 (d) 0 (f) 4

$$\lim_{n\to\infty} \left| \frac{(2x+8)^{n+1}}{(n+1)!} \cdot \frac{n!}{(2x+8)^n} \right| = \lim_{n\to\infty} \left| \frac{(2x+8)^{n+1}}{(2x+8)^n} \cdot \frac{n!}{(n+1)!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{2x+8}{n+1} \right| = 0 < 1$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{(2x+8)^n}{n!} \text{ converges for all } x.$$

$$\Rightarrow I = (-\infty, \infty)$$

Q. How can we tell what a power series converges to?

We can't! (Usually)

Q. How can we convert familiar functions to power series?

① Compare to something we know (e.g. $\frac{1}{1-x}$)

② Something else?

Ex 10 Find the power series centered at $a = 9$ which converges to $f(x) = \frac{2}{1-x}$ and identify the interval of convergence of the power series.

$$f(x) = \frac{2}{1-x} = \frac{2}{1-(x-9+9)} = \frac{2}{1-(x-9)-9}$$

$$= \frac{2}{-8-(x-9)} = \frac{2}{-8}\,\frac{1}{1-\left(\frac{x-9}{-8}\right)}$$

$u = \frac{x-9}{-8}$

$$= -\frac{1}{4}\cdot\frac{1}{1-u} = -\frac{1}{4}\cdot\sum_{n=0}^{\infty} u^n$$

$$= -\frac{1}{4}\cdot\sum_{n=0}^{\infty}\left(\frac{x-9}{-8}\right)^n = \sum_{n=0}^{\infty}\frac{-(x-9)^n}{4\cdot 8^n}$$

Converges if and only if $|u| < 1 \longleftrightarrow \left|\frac{x-9}{-8}\right| < 1$

$\longleftrightarrow |x-9| < 8 \longleftrightarrow$ x is within 8 of 9

$\longleftrightarrow x \in (1, 17) = I$

Ex 11 Find the power series for $f(x) = \frac{x}{4+x^2}$ centered at $a = 0$ and identify the interval of convergence.

Q. What's a helpful form for $f(x)$?

(a) $f(x) = x \cdot \dfrac{1}{4+x^2}$

(b) $f(x) = \dfrac{x}{4} \cdot \dfrac{1}{1+\frac{x^2}{4}}$

(c) $f(x) = \dfrac{x}{-4} \cdot \dfrac{1}{-1-\frac{x^2}{4}}$

(d) $f(x) = \dfrac{x}{4} \cdot \dfrac{1}{1-(-\frac{x^2}{4})}$

$$f(x) = \frac{x}{4} \cdot \frac{1}{1-\left(-\frac{x^2}{4}\right)} = \frac{x}{4} \sum_{n=0}^{\infty} u^n$$

$u = -\frac{x^2}{4}$

$$= \frac{x}{4} \sum_{n=0}^{\infty} \left(-\frac{x^2}{4}\right)^n = \frac{x}{4} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{4^{n+1}}$$

convergence if and only if $|u| < 1$

$\leftrightarrow \left|-\frac{x^2}{4}\right| < 1 \leftrightarrow \frac{x^2}{4} < 1$

$\leftrightarrow x^2 < 4 \leftrightarrow |x| < 2$

$\leftrightarrow \boxed{x \in (-2, 2) = I}$

Ex 12 Find the power series for $f(x) = \frac{1}{x}$ centered at $a = 1$ and identify the interval of convergence.

Q. What's the most helpful form for $f(x)$?

(a) $f(x) = \frac{1}{x}$

(b) $f(x) = \frac{1}{1-(1-x)}$

(c) $f(x) = \frac{1}{1-(-(x-1))}$

(d) $f(x) = x^{-1}$

$$f(x) = \frac{1}{1-(-(x-1))} = \sum_{n=0}^{\infty} u^n$$

u

$$= \sum_{n=0}^{\infty} (-(x-1))^n$$

$$= \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

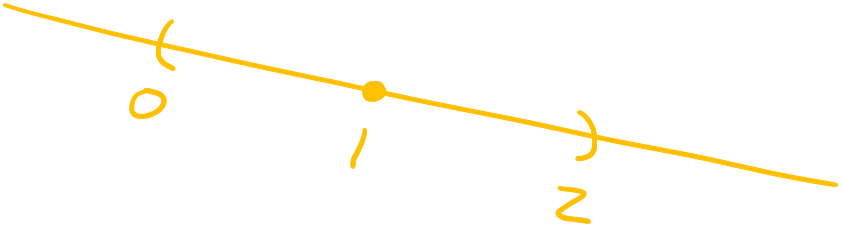
Fact: $|-t| = |t|$

converges iff $|u| < 1$

$\leftrightarrow |-(x-1)| < 1$

$\leftrightarrow |x-1| < 1$

$\leftrightarrow x \in (0, 2)$



Ex 13 Find a power series for $f(x) = \dfrac{9-3x}{x^2-6x}$.

Q. What technique do you think will be helpful here?

(a) Factor the denominator
(b) Partial fraction decomposition
(c) Complete the square in the denominator
(d) Long divide the numerator and the denominator

$$f(x) = \frac{9-3x}{x(x-6)} = \frac{3(3-x)}{x(x-6)}$$

$$= \frac{3(3-x)}{x} \cdot \frac{1}{x-6}$$

$$= \frac{3(3-x)}{x} \cdot \frac{1}{-6} \quad \frac{1}{1-\frac{x}{6}}$$

$$= \frac{3(3-x)}{x} \cdot \frac{1}{-6} \quad \sum \left(\frac{x}{6}\right)^n$$

$$(x-a)^2 = x^2 - 2ax + a^2$$

$$f(x) = \frac{9-3x}{x^2-6x+9-9}$$

$$= \frac{9-3x}{(x-3)^2-9} = \frac{3(3-x)}{-1} \cdot \frac{1}{9-(x-3)^2}$$

$$= \frac{3(3-x)}{-9} \cdot \frac{1}{1-\frac{(x-3)^2}{9}}$$

$$= \frac{-3(x-3)}{-9} \cdot \sum_{n=0}^{\infty} \left(\frac{(x-3)^2}{9}\right)^n$$

$$= \frac{(x-3)}{3} \sum_{n=0}^{\infty} \frac{(x-3)^{2n}}{9^n}$$

$$= \sum_{n=0}^{\infty} \frac{(x-3)^{2n+1}}{3 \cdot 9^n}$$

Converges iff $\left|\frac{(x-3)^2}{9}\right| < 1$

$\leftrightarrow \frac{(x-3)^2}{9} < 1 \leftrightarrow (x-3)^2 < 9$

$\leftrightarrow |x-3| < 3$

$\leftrightarrow x \in (0, 6)$

Power series: $\sum c_n (x-a)^n$